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Extensions and Fixed Points of Contractive Maps in $\mathbb{R}^n$

D. S. BRIDGES

Department of Mathematics and Statistics, University of Waikato, Hamilton, New Zealand

AND

F. RICHMAN, W. H. JULIAN, AND R. MINES

Department of Mathematical Sciences, New Mexico State University, Las Cruces, New Mexico 88003

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This paper, which is written within the framework of Bishop's constructive mathematics, deals with the construction of the fixed point ξ of a contractive self-map f of $\mathbb{R}^n$, and with the rate at which the sequence $(f^n(x))$ converges to ξ for any x in $\mathbb{R}^n$. It also discusses contractive extensions of contractive mappings on compact subsets of $\mathbb{R}^n$, and almost uniform contractions of complete metric spaces.

1. INTRODUCTION

To set the scene for this paper, which is written within the framework of Bishop's constructive mathematics and arose from an enquiry into the constructive content of Edelstein's extension of the Banach contraction mapping theorem [Ed, Thm 1], we define a mapping f from a metric space X into a metric space Y to be

contractive if $\rho(f(x), f(y)) < \rho(x, y)$ for all distinct points x, y of X,

and

nonexpansive if $\rho(f(x), f(y)) \leqslant \rho(x, y)$ for all x, y in X.[1]

[1] We use ρ to denote the metric on any metric space; points x and y of a metric space are *distinct* if $\rho(x, y) > 0$, in which case we write $x \neq y$.

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We say that a mapping f from a set S into itself is a *self-map* of S; for each x in S we define the sequence of *iterates* of f as follows:

$$\begin{aligned} f^k(x) &= x && \text{if} \quad k=0, \\ &= f(f^{k-1}(x)) && \text{if} \quad k \geqslant 1. \end{aligned}$$

Edelstein's theorem states that a contractive self-map f of a compact metric space X has a unique fixed point ξ in X, and that for each x in X the sequence of iterates $(f^k(x))$ converges to ξ. A constructive proof of the first part of this theorem would embody an algorithm to convert the data describing X and f into arbitrarily close approximations to the fixed point ξ; a constructive proof of the second part would provide a rate of convergence for the sequence $(f^k(x))$. We know of no classical proof of either part that is constructive in this sense.

Consider, for example, perhaps the simplest of all classical proofs of the first part of Edelstein's theorem: if ξ is the point of X where the continuous mapping $x \mapsto \rho(x, f(x))$ attains its infimum, then, since the assumption $\xi \neq f(\xi)$ leads to the contradiction $\rho(f(\xi), f^2(\xi)) < \rho(\xi, f(\xi))$, we must have $\xi = f(\xi)$; the contractive property of f ensures that ξ is its unique fixed point. From a constructive point of view, this proof is inadequate because *there is no algorithm which, applied to a continuous function on a compact metric space, will compute a point where that function attains its infimum.*[2] Indeed, within the framework of recursive analysis we can construct a uniformly continuous, everywhere positive self-map f of the recursive interval $[0, 1]$ such that $\inf f = 0$ [JR; BR, Chap. 6, 2.9].

In Section 2 of this paper we present an algorithm for the computation of the fixed point of a contractive self-map of a compact convex subset of $\mathbb{R}^n$. We then show how to extend a contractive self-map of a compact, not necessarily convex, subset of $\mathbb{R}$ to one on the whole of $\mathbb{R}$, and apply that extension theorem to the construction of the fixed point of a contractive self-map of a compact subset of $\mathbb{R}$; we also prove that the sequence of iterates converges to the fixed point uniformly on compact sets.

In Section 3 we prove that if f is a bounded contractive self-map of $\mathbb{R}^n$, and $x \in \mathbb{R}^n$, then x is close to the fixed point of f whenever $f(x)$ is close to x. We use this result to construct a recursive example of a contractive map of a compact set $K \subset \mathbb{R}^2$ into $\mathbb{R}^2$ that cannot be extended to a bounded contractive self-map of $\mathbb{R}^2$.

In Section 4 we introduce *almost uniform contractions* and prove some results, analogous to those in our earlier sections, about their fixed points.

[2] Note that there is no problem computing the infimum, as long as we know that f is uniformly continuous on the compact metric space X.

In Section 5 we discuss the convergence of the sequence of iterates of a bounded contractive self-map of $\mathbb{R}^2$.

It is reasonable to ask whether our work conveys any more information than that of Scarf [Sc] on the computation of fixed points of a continuous self-map f of the closed unit ball in $\mathbb{R}^n$. Algorithms such as Scarf's do not compute fixed points: they cannot, since the classical Brouwer fixed-point theorem is not constructive, even in the case $n = 1$ [BR, Chap. 3, 2.4]; they produce, for each $\varepsilon > 0$, an *ε-approximate fixed point* of f—that is, a point x such that $\|f(x) - x\| < \varepsilon$. In the case where f is contractive, it is simple to prove classically, by a contradiction argument, that if, for each n, x_n is a $1/n$-approximate fixed point of f, then (x_n) converges to the fixed point ξ of f; however *such an argument does not provide the rate at which the sequence of approximate fixed points converges to ξ*. In contrast, the work below does provide such a rate of convergence; see Proposition 3.2.

Although we have chosen to work within Bishop's constructive mathematics, all the results and techniques we use are fully compatible with Church's thesis. Our work can therefore be regarded as part of a recursive development of fixed-point theory.

We refer the reader to [BB, BR, Ku] for background material in constructive mathematics. In particular, we will use results from [BB, Chaps. 2 and 4] without specific reference.

2. Fixed Points of Contractive Mappings in $\mathbb{R}^n$

We begin with a simple lemma dealing with the uniqueness of fixed points of contractive mappings.

(2.1) Lemma. *If f is a contractive self-map of a metric space X, and if x, y are distinct points of X, then either $f(x) \neq x$ or $f(y) \neq y$.*

Proof. This is an immediate consequence of the inequalities

$$\rho(x, f(x)) + \rho(y, f(y)) \geqslant \rho(x, y) - \rho(f(x), f(y)) > 0. \quad \blacksquare$$

A mapping $\varphi\colon X \to Y$ of metric spaces is *locally nonconstant* if for each $x \in X$ and each neighbourhood U of x, there exists $x' \in U$ such that $f(x) \neq f(x')$. Although the intermediate value theorem cannot be proved constructively for all continuous functions, it does hold for all locally nonconstant, pointwise continuous functions from an interval to $\mathbb{R}$ [BR, Chap. 2, Sect. 2].

The following theorem will lead to the Brouwer Fixed-Point Theorem for contractive maps.

(2.2) THEOREM. *A bounded contractive self-map of $\mathbb{R}^n$ has a unique fixed point.*

Proof. Given a bounded contractive self-map f of $\mathbb{R}^n$, we proceed by induction on n. The theorem is trivial if $n=0$. Suppose we have proved it for $n=0, ..., k-1$, and consider a bounded contractive self-map f of $\mathbb{R}^k$. Let $\mathbf{v}$ be a unit vector in $\mathbb{R}^k$. For each real number r let H_r be the hyperplane in $\mathbb{R}^k$ through $r\mathbf{v}$ and perpendicular to $\mathbf{v}$, and let π_r be the orthogonal projection of $\mathbb{R}^k$ onto H_r. Then the restriction of $\pi_r \circ f$ to H_r is a bounded contractive self-map of H_r, and so, by our induction hypothesis, has a unique fixed point z_r in H_r. Since $\pi_r(f(z_r)-z_r)=0$, there is a unique real number $\varphi(r)$ such that $f(z_r)-z_r=\varphi(r)\,\mathbf{v}$. The mapping $\varphi\colon \mathbb{R}\to\mathbb{R}$ so defined is uniformly continuous, since for all r, s in $\mathbb{R}$,

$$\begin{aligned}
0 &\geqslant |f(z_r)-f(z_s)|^2-|z_r-z_s|^2\\
&=\langle f(z_r)-f(z_s)+z_r-z_s, f(z_r)-f(z_s)-z_r+z_s\rangle\\
&=|\varphi(r)-\varphi(s)|^2+2\langle z_r-z_s, (\varphi(r)-\varphi(s))\,\mathbf{v}\rangle\\
&\geqslant |\varphi(r)-\varphi(s)|^2-2\,|r-s|\,|\varphi(r)-\varphi(s)|\\
&=|\varphi(r)-\varphi(s)|\,(|\varphi(r)-\varphi(s)|-2\,|r-s|)
\end{aligned}$$

and therefore

$$|\varphi(r)-\varphi(s)|<2\,|r-s|.$$

Since f is bounded, φ takes both positive and negative values; also, by (2.1), φ is locally nonconstant. Hence, by [BR, Chap. 3, 2.5], there exists r such that $\varphi(r)=0$. Thus z_r is a fixed point of f. Reference to (2.1) shows that z_r is the unique fixed point of f. ∎

We now recall the most general constructive version of the

BROUWER FIXED-POINT THEOREM [RBCM, Thm 11]. *If f is a uniformly continuous self-map of the closed unit ball in $\mathbb{R}^n$, then for each $\varepsilon>0$ there exists x such that $\|f(x)-x\|<\varepsilon$.*

The case $n=1$ is discussed in [BR, Chap. 3, Sect. 2; Ku, p. 190, Cor. 1], where it is proved that the intermediate value theorem is false in recursive constructive mathematics. Within recursive constructive mathematics, there exists a pointwise continuous self-map of the closed unit ball in $\mathbb{R}^n$ that moves each point of the ball a distance $\frac{1}{5}$ [Be, Chap. IV, 10.1, Corollary].

We now have the Brouwer Fixed-Point Theorem for contractive maps:

(2.3) COROLLARY. *A contractive self-map of a compact convex subset of $\mathbb{R}^n$ has a unique fixed point.*

Proof. Let f be a contractive self-map of a compact convex set $K \subseteq \mathbb{R}^n$, and let π be the projection of $\mathbb{R}^n$ onto K. By (2.2), the contractive map $f \circ \pi$ has a unique fixed point ξ, which belongs to K and is the unique fixed point of f. ∎

We will prove shortly that in the case $n = 1$ the above corollary holds with "convex" replaced by "nonvoid."

The *metric complement* of a located subset K of $\mathbb{R}$ is the set

$$K' \equiv \{x \in \mathbb{R} : \rho(x, K) > 0\}.$$

If K is closed in $\mathbb{R}$, then $x \in K'$ whenever $x \neq y$ for all $y \in K$ [BB, Chap. 2, 3.8]. If $K \subset \mathbb{R}$ is compact, then K' can be written in the form

$$K' = \cup \{I_x : x \in K'\},$$

where for each x in K', I_x is an open interval—called a *gap* of K—of one of three types:

$$I_x = (-\infty, m), \qquad \text{where} \quad m \equiv \inf K;$$

$$I_x = (M, \infty), \qquad \text{where} \quad M \equiv \sup K;$$

$$I_x = (a, b) \qquad \text{for some } a, b \text{ in } K \text{ with } a < b.$$

To see this, consider any x in K', and set $r \equiv \rho(x, K) > 0$. Either $x < m$, in which case we take $I_x \equiv (-\infty, m)$; or else $x > m - r$ and therefore $x \geqslant m$. In the latter case, either $x > M$, when we take $I_x \equiv (M, \infty)$; or else $x < M + r$ and therefore $x \leqslant M$. So we may assume that $m \leqslant x \leqslant M$. Then the sets

$$K^- \equiv \{k \in K : k \leqslant x - r\} = \{k \in K : k < x\},$$

$$K^+ \equiv \{k \in K : k \geqslant x + r\} = \{k \in K : k > x\}$$

are nonvoid and compact, and we take $I_x \equiv (a, b)$, where $a \equiv \sup K^-$ and $b \equiv \inf K^+$.

The expression of the metric complement as a union of gaps is used in the proof of our next result, an extension theorem for contractive maps on compact subsets of $\mathbb{R}$.

(2.4) THEOREM. *Let K be a nonvoid compact subset of $\mathbb{R}$, and f a contractive mapping of K into a convex subset S of a normed linear space V. Then there exists a contractive mapping $F: \mathbb{R} \to S$ such that*

(i) *$F(x) = f(x)$ for all x in K, and*

(ii) *$F(x) \neq x$ for all x such that $\rho(x, K) > 0$.*

Proof. Let D be the union of K and its metric complement, and define a map $F: D \to V$ as follows. If $x \in K$, then $F(x) \equiv f(x)$; if $x < \inf K$, then $F(x) \equiv f(\inf K)$; if $x > \sup K$, then $F(x) \equiv f(\sup K)$; if x belongs to a bounded gap $I_x \equiv (a, b)$ in K, then $x = \lambda a + (1 - \lambda) b$ for a unique λ in $[0, 1]$, and we define

$$F(x) \equiv \lambda f(a) + (1 - \lambda) f(b).$$

Since S is convex, F maps D into S. By considering all the possibilities for points x, y of D with $x < y$, it is easy to show that F is contractive on D. For example, if x and y both belong to a bounded gap (a, b) in K, then there exist distinct λ and μ such that $x = \lambda a + (1 - \lambda) b$ and $y = \mu a + (1 - \mu) b$; so

$$\begin{aligned} |F(x) - F(y)| &= |\lambda - \mu| \, |f(a) - f(b)| \\ &< |\lambda - \mu| \, |a - b| = |x - y|; \end{aligned}$$

whereas if x is in a bounded gap (a, b), and y is in a bounded gap (c, d), with $b \leqslant c$, then

$$\begin{aligned} |F(x) - F(y)| &\leqslant |F(x) - f(b)| + |f(b) - f(c)| + |f(c) - F(y)| \\ &< (b - x) + (c - b) + (y - c) \\ &= y - x = |x - y|. \end{aligned}$$

Being contractive, F is uniformly continuous on the dense subset D of $\mathbb{R}$; it therefore extends uniquely to a nonexpansive map F on $\mathbb{R}$. Since D is order dense in $\mathbb{R}$—that is, for all a, b in $\mathbb{R}$ with $a < b$ there exists x in D such that $a < x < b$—it readily follows that F is actually contractive on $\mathbb{R}$. ∎

We cannot remove from Theorem 2.4 the hypothesis that K is compact. Consider the contractive mapping f defined on $(-\infty, -1) \cup (1, \infty)$ by

$$\begin{aligned} f(x) &= -1 \qquad \text{if} \quad x < -1, \\ &= 1 \qquad \text{if} \quad x > 1. \end{aligned}$$

Any continuous extension of f to $\mathbb{R}$ has both -1 and 1 as fixed points, and so, by Theorem 2.2, is not contractive.

Theorem 2.4 will enable us to strengthen Corollary 2.3 in the case $n = 1$. It is convenient first to prove the following lemma.

(2.5) LEMMA. *Let X be a totally bounded metric space, and f a self-map of X that has a fixed point ξ. Suppose that for each x in X the sequence*

$(f^n(x))$ converges to ξ. Then the sequence (f^n) converges to ξ uniformly on X.

Proof. Given $\varepsilon > 0$, let $\{x_1, ..., x_\nu\}$ be an $\varepsilon/2$-approximation to X. There exists a positive integer N such that $\rho(f^N(x_i), \xi) < \varepsilon/2$ for each i. Given x in X, choose i such that $\rho(x, x_i) < \varepsilon/2$. Then, as f is contractive,

$$\begin{aligned}\rho(f^N(x), \xi) &\leqslant \rho(f^N(x), f^N(x_i)) + \rho(f^N(x_i), \xi)\\ &\leqslant \rho(x, x_i) + \varepsilon/2 < \varepsilon,\end{aligned}$$

and therefore $\rho(f^n(x), \xi) < \varepsilon$ for all $n \geqslant N$. ∎

(2.6) THEOREM. *Let K be a nonvoid compact subset of $\mathbb{R}$, and $f\colon K \to K$ a contractive mapping. Then*

(i) *f has a unique fixed point ξ; and*

(ii) *the sequence (f^n) converges to ξ uniformly on K.*

Proof. Construct a contractive mapping $F\colon \mathbb{R} \to [\inf K, \sup K]$ as in (2.4). By (2.2), F has a unique fixed point ξ in $\mathbb{R}$. It follows from (2.4), (ii), that $\rho(\xi, K) = 0$; whence $\xi \in K$ and therefore $f(\xi) = F(\xi) = \xi$. By (2.1), ξ is the unique fixed point of f. This completes the proof of (i).

In view of (2.5), to prove (ii) it will suffice to show that $f^n(x) \to \xi$ as $n \to \infty$ for each x in K. Since f is contractive, we need only prove that for each $\varepsilon > 0$ there exists n such that $|f^n(x) - \xi| < 3\varepsilon$. As K is totally bounded, there exist j, k such that $j > k$ and $|f^j(x) - f^k(x)| < \varepsilon$; so, replacing x by $f^k(x)$ and f by f^{j-k}, we may assume that $|f(x) - x| < \varepsilon$. Either $|f(x) - \xi| < 3\varepsilon$ and there is nothing further to prove, or else $|f(x) - \xi| > 2\varepsilon$. In the latter case, we may assume that $f(x) > \xi + 2\varepsilon$; then $x > \xi + \varepsilon$, so, as f is contractive, $x - \xi > f(x) - \xi$ and therefore $x > f(x)$. Setting

$$\delta \equiv \xi + \varepsilon - F(\xi + \varepsilon) > 0,$$

choose a positive integer N such that $N\delta > x - \xi - \varepsilon$. If $y > \xi + \varepsilon$, then $y - (\xi + \varepsilon) > F(y) - F(\xi + \varepsilon)$ and therefore $y - F(y) > \delta$. Hence for some integer n with $2 \leqslant n \leqslant N$, we have $f^{n-1}(x) > \xi + \varepsilon$ and $f^n(x) < \xi + 2\varepsilon$. Also, as f is contractive and $f^{n-1}(x) \neq \xi$,

$$0 < f^{n-1}(x) - f^n(x) < x - f(x) < \varepsilon,$$

so $f^n(x) > f^{n-1}(x) - \varepsilon > \xi$. Hence $|f^n(x) - \xi| < 2\varepsilon$. ∎

(2.7) COROLLARY. *If f is a bounded contractive self-map of $\mathbb{R}$ with fixed point ξ, then the sequence $(f^n(x))$ converges to ξ uniformly on compact subsets of $\mathbb{R}$.*

Proof. Choose $r>0$ such that $|x| \leqslant r$ and f maps $\mathbb{R}$ into $K \equiv [-r, r]$, and apply (2.6) with f replaced by its restriction to K. ∎

3. Approximate Fixed Points of Contractive Mappings

Our next lemma will enable us to prove that under the hypotheses of Theorem 2.2, if $f(x)$ is close to x, then x is close to the fixed point of f.

(3.1) Lemma. *Let f be a bounded contractive self-map of $\mathbb{R}^n$ with fixed point ξ, H a hyperplane in $\mathbb{R}^n$, and z a point of H such that $f(z)-z$ is non-zero and orthogonal to H. Then*

(i) *$f(z)$ and ξ are on the same side of H;*

(ii) *there exists $s>0$ such that $\|x-f(z)\| - \|f(x)-f(z)\| \geqslant s$ for each point x on the side of H opposite to $f(z)$.*

Proof. Translating the origin if necessary, we may assume that $z=0$. As f is continuous, ξ is fixed, and $f(0) \neq 0$, we have $\xi \neq 0$. Write $\xi = y + tf(0)$, where $y \in H$ and $t \in \mathbb{R}$. Then the inequality $\|\xi - f(0)\|^2 < \|\xi\|^2$ leads to

$$2t \, \|f(0)\|^2 = 2\langle \xi, f(0)\rangle > \|f(0)\|^2;$$

so $2t>1$, $t>\frac{1}{2}$, and therefore ξ and $f(0)$ are on the same side of H.

To prove (ii), choose $r>0$ such that $\|f(x)-f(0)\| \leqslant r$ for all x in $\mathbb{R}^n$. Set

$$s \equiv \min\{r, \|f(0)\|^2/(6r + \|f(0)\|)\},$$

and consider any x on the side of H opposite to $f(0)$. Then $\langle x, f(0)\rangle \leqslant 0$, so

$$\|x-f(0)\|^2 - \|x\|^2 = \|f(0)\|^2 - 2\langle x, f(0)\rangle \geqslant \|f(0)\|^2.$$

Thus if $\|x-f(0)\| < 3r$, then

$$\begin{aligned}\|f(0)\|^2 &\leqslant (\|x-f(0)\| + \|x\|)(\|x-f(0)\| - \|x\|)\\ &\leqslant (2\,\|x-f(0)\| + \|f(0)\|)(\|x-f(0)\| - \|x\|)\\ &\leqslant (6r + \|f(0)\|)(\|x-f(0)\| - \|x\|).\end{aligned}$$

As f is contractive, it follows that

$$\|x-f(0)\| - \|f(x)-f(0)\| \geqslant \|x-f(0)\| - \|x\| \geqslant s.$$

On the other hand, if $\|x-f(0)\| > 2r$, then

$$\|x-f(0)\| - \|f(x)-f(0)\| > 2r - r \geqslant s. \quad ∎$$

(3.2) PROPOSITION. *Let f be a bounded contractive self-map of $\mathbb{R}^n$, and let ξ be the unique fixed point of f. For each $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|x - f(x)\| \leqslant \delta$, then $\|x - \xi\| \leqslant \varepsilon$.*

Proof. We first prove that if H is a hyperplane and $\rho(\xi, H) > 0$, then there exists $z \in H$ such that $f(z) - z$ is nonzero and orthogonal to H. If π is the orthogonal projection of $\mathbb{R}^n$ onto H, then the restriction of $\pi \circ f$ to H is a bounded contractive self-map of H and so, by (2.2), has a unique fixed point z. Certainly, $f(z) - z$ is orthogonal to H; on the other hand, as $z \neq \xi$, it follows from (2.1) that $f(z) - z \neq 0$.

Now let C be a nondegenerate hypercube centered at ξ such that $\|x - \xi\| \leqslant \varepsilon$ for each x in C. Applying first the preceding part of this proof, and then (3.1), to each hyperplane that contains a face of C, compute $\delta > 0$ such that if $\rho(x, C) > 0$, then $\|f(x) - x\| > \delta$. So if $\|f(x) - x\| \leqslant \delta$, then $x \in C$ and therefore $\|x - \xi\| \leqslant \varepsilon$. ∎

A classical theorem of Kirszbraun states that a nonexpansive map of a subset X of $\mathbb{R}^n$ into a normed linear space V extends to a nonexpansive mapping of $\mathbb{R}^n$ into V [Is, pp. 187, 77–108]. Theorem 2.4 says that "nonexpansive" can be replaced by "contractive" in Kirszbraun's theorem provided X is compact and $n = 1$. The example following Theorem 2.4 shows that we cannot remove the compactness of X from the hypotheses of that analogue of Kirszbraun's theorem; the following proposition shows that, in our constructive framework, we cannot replace $\mathbb{R}$ by $\mathbb{R}^2$ in that analogue.[3]

(3.3) PROPOSITION. *Under Church's thesis, there exist a compact subset K of $\mathbb{R}^2$, a sequence (x_n) in K, and a contractive homeomorphism f of K onto the boundary of the closed unit disc D in $\mathbb{R}^2$, such that $f(x) \neq x$ for each x in K, and $\|f(x_n) - x_n\| \to 0$ as $n \to \infty$.*

Proof. Assuming Church's thesis, we can construct a uniformly continuous function ψ from the unit circle in $\mathbb{R}^2$ into $(0, 1]$ such that $\inf \psi = 0$ [JR; BR, Chap. 6, 2.9]. The mapping $z \mapsto (1 + \psi(z))\, z$ is a homeomorphism of the boundary ∂D of D onto the set

$$K \equiv \{(1 + \psi(z))\, z : z \in \mathbb{R}^2, \|z\| = 1\},$$

which is therefore compact; the inverse f of this mapping is the projection, a contractive map, of K onto D. For each $x \in K$ we have $\|f(x) - x\| = \psi(\|x\|^{-1} x) > 0$. To complete the proof, construct a sequence (z_n) in ∂D such that $\psi(z_n) \to 0$ as $n \to \infty$, and set $x_n \equiv (1 + \psi(z_n))\, z_n$ for each n. ∎

[3] We do not know if it is true classically that a contractive mapping of a compact set $K \subset \mathbb{R}^n$ into $\mathbb{R}^n$ extends to a contractive mapping of $\mathbb{R}^n$ into itself.

(3.4) COROLLARY. *Under Church's thesis, there exist a compact set $K \subseteq \mathbb{R}^2$ and a contractive mapping of K onto the boundary of the closed unit disc D in $\mathbb{R}^2$ that cannot be extended to a contractive mapping of $\mathbb{R}^2$ into D.*

Proof. Let K, f, and (x_n) be as in (3.3), and suppose that f extends to a contractive mapping F of $\mathbb{R}^2$ into D. By (2.2), F has a unique fixed point ξ in $\mathbb{R}^2$. Proposition 3.2 shows that $x_n \to \xi$ as $n \to \infty$. Since K is closed and ψ is continuous, it follows that $\xi \in K$ and $\psi(\xi) = 0$, a contradiction. ∎

4. FIXED POINTS OF ALMOST UNIFORM CONTRACTIONS

Let X, Y be metric spaces, and f a mapping of X into Y; we say that f is an *almost uniform contraction* if for each $\varepsilon > 0$ there exists $r \in (0, 1)$ such that $\rho(f(x), f(y)) \leqslant r\rho(x, y)$ whenever x, y belong to X and $\rho(x, y) \geqslant \varepsilon$. Clearly, an almost uniform contraction is contractive. A simple contradiction argument using sequential compactness shows classically that a contractive map on a compact metric space is an almost uniform contraction. Our next two lemmas enable us to prove this constructively when the compact space is a subset of $\mathbb{R}$.

(4.1) LEMMA. *Let K be a nonvoid compact subset of $\mathbb{R}$, and $\varepsilon > 0$. Then there exists a finitely enumerable subset F of $K \times K$ such that*

(i) *$a < b$ whenever $(a, b) \in F$; and*

(ii) *if $x, y \in K$ and $x \leqslant y - \varepsilon$, then $x \leqslant a < b \leqslant y$ for some $(a, b) \in F$.*

Proof. Let $m \equiv \inf K$ and $M \equiv \sup K$. If $M - m < \varepsilon$, take $F \equiv \phi$. If $M - m > 0$, choose a positive integer $n > 2(M - m)/\varepsilon$, and let $t_i \equiv m + i(M - m)/n$ $(i = 1, \ldots, n-1)$. Then choose a positive number $\delta < (M - m)/2n$ so that the sets

$$K_i^+ = K \cap [t_i + \delta, M] \qquad \text{and} \qquad K_i^- \equiv K \cap [m, t_i - \delta]$$

are compact for $i = 1, \ldots, n-1$; for each such i, let $a_i \equiv \sup K_i^-$ and $b_i \equiv \inf K_i^+$. Then

$$F \equiv \{(a_i, b_i) : i = 1, \ldots, n-1\}$$

is a finitely enumerable subset of $K \times K$ satisfying (i). If $x, y \in K$ and $x \leqslant y - \varepsilon$, then there exists i such that $x + \varepsilon/4 < t_i < y - \varepsilon/4$; whence $x \leqslant a_i < b_i \leqslant y$. ∎

(4.2) LEMMA. *Let X be a subset of $\mathbb{R}$, f a contractive map from X into a metric space Y, and a, b, x, y points of X such that $x \leqslant a < b \leqslant y$. Then*

$$(y-x) - \rho(f(x), f(y)) \geqslant (b-a) - \rho(f(a), f(b)),$$

with strict inequality if either $x < a$ or $y > b$.

Proof. Since f is contractive, we have

$$\begin{aligned}\rho(f(x), f(y)) &\leqslant \rho(f(x), f(a)) + \rho(f(a), f(b)) + \rho(f(b), f(y)) \\ &\leqslant (a-x) + \rho(f(a), f(b)) + (y-b),\end{aligned}$$

with strict inequality at the last step if either $x < a$ or $y > b$. The desired conclusion follows immediately. ∎

(4.3) PROPOSITION. *A contractive map of a compact subset of $\mathbb{R}$ into a metric space is an almost uniform contraction.*

Proof. Let f be a contractive map of a compact set $K \subset \mathbb{R}$ into a metric space X. Choose a positive integer n such that $|x| \leqslant n$ for all x in K. Given $\varepsilon > 0$, construct a finitely enumerable subset F of $K \times K$ as in (4.1), and define

$$\begin{aligned}\delta &\equiv \inf\{b - a - |f(b) - f(a)| : (a, b) \in F\}, \\ r &\equiv 1 - \delta/2n.\end{aligned}$$

Note that $\delta < 2n$, so $0 < r < 1$. By (4.2), if $x, y \in K$ and $|x-y| \geqslant \varepsilon$, then

$$\begin{aligned}\rho(f(x), f(y)) &\leqslant |x-y| - \delta \\ &\leqslant |x-y|\,(1 - \delta/2n) \\ &= r\,|x-y|. \quad ∎\end{aligned}$$

(4.4) LEMMA. *Let f be an almost uniform contraction of a metric space X into itself. For each $\varepsilon > 0$ there exists $\delta > 0$ such that if $x, y \in X$, $\rho(x, f(x)) \leqslant \delta$, and $\rho(y, f(y)) \leqslant \delta$, then $\rho(x, y) \leqslant \varepsilon$.*

Proof. Given $\varepsilon > 0$, let $r \in (0, 1)$ be as in the definition of "almost uniform contraction," set $\delta \equiv \varepsilon(1-r)/2$, and consider points x, y of X such that $\rho(x, f(x)) \leqslant \delta$ and $\rho(y, f(y)) \leqslant \delta$. If $\rho(x, y) > \varepsilon$, then

$$\begin{aligned}(1-r)\,\rho(x, y) &\leqslant \rho(x, y) - \rho(f(x), f(y)) \\ &< \rho(x, f(x)) + \rho(y, f(y)) \\ &\leqslant 2\delta = \varepsilon(1-r);\end{aligned}$$

whence $\rho(x, y) < \varepsilon$, a contradiction from which the desired conclusion follows. ∎

Proposition 4.3 does not generalize to cover compact subsets of $\mathbb{R}^n$:

(4.5) PROPOSITION. *Under Church's thesis, there exist a compact subset K of $\mathbb{R}^2$ and a contractive map of K into $\mathbb{R}^2$ that is not an almost uniform contraction.*

Proof. Let $K, f,$ and (x_n) be as in (3.3), and suppose that f is an almost uniform contraction. It follows from (4.4) that (x_n) is a Cauchy sequence in K and so converges to a limit $\xi \in K$. As ψ is continuous, $\psi(\xi) = 0$, a contradiction. ∎

For almost uniform contractions of a complete metric space into itself, fixed points are relatively easy to compute.

(4.6) THEOREM. *Let f be an almost uniform contraction of a complete metric space X into itself. Then*

(i) *f has a unique fixed point ξ in X; and*

(ii) *the sequence $(f^n(x))$ converges to ξ uniformly on each bounded subset of X.*

Proof. Let $B \subset X$ be bounded. We may assume that B is nonvoid. Let $b \in B$, and choose $c > 0$ such that

$$2\rho(x, b) + \rho(b, f(b)) \leqslant c$$

for all x in B. Then for all such x we have

$$\begin{aligned}\rho(x, f(x) &\leqslant \rho(x, b) + \rho(f(x), f(b)) + \rho(f(b), b)\\ &\leqslant 2\rho(x, b) + \rho(b, f(b)) \leqslant c.\end{aligned}$$

Given $\varepsilon > 0$, let $r > 0$ be as in the definition of "almost uniform contraction," and choose a positive integer N such that

$$r^N c < \varepsilon' \equiv \varepsilon(1 - r)/2;$$

then $r^N \rho(x, f(x)) < \varepsilon'$ for all x in B.

Given any x in B, write $x_n \equiv f^n(x)$ for each n. Let $n \geqslant N$, and suppose that $\rho(x_n, x_{n+1}) > \varepsilon'$. Then $\rho(x_k, x_{k+1}) > \varepsilon'$, and therefore $\rho(x_{k+1}, x_{k+2}) < r\rho(x_k, x_{k+1})$, for $k = 0, 1, ..., N$; whence $\rho(x_N, f(x_N)) < r^N \rho(x_0, x_1) < \varepsilon'$, a contradiction. Thus $\rho(x_n, x_{n+1}) \leqslant \varepsilon'$ for all $n \geqslant N$.

Now consider any $m, n \geqslant N$, and suppose that $\rho(x_m, x_n) > \varepsilon$. We have

$$\rho(x_m, x_n) - \rho(x_n, x_{n+1}) - \rho(x_m, x_{m+1}) \leqslant \rho(x_{m+1}, x_{n+1}) \leqslant r\rho(x_m, x_n);$$

whence

$$(1-r)\,\rho(x_m, x_n) \leqslant \rho(x_n, x_{n+1}) + \rho(x_m, x_{m+1}) < 2\varepsilon' = (1-r)\,\varepsilon$$

and therefore $\rho(x_m, x_n) < \varepsilon$, a contradiction. Thus $\rho(x_m, x_n) \leqslant \varepsilon$ whenever $m, n \geqslant N$; so (x_n) is a Cauchy sequence in X. As X is complete, (x_n) converges to a limit ξ in X, such that $\rho(x_n, \xi) \leqslant \varepsilon$ for all $n \geqslant N$. Clearly, ξ is a fixed point of f. But, by (2.1), ξ is the unique fixed point of f. Hence for each x in B and all $n \geqslant N$, $\rho(f^n(x), \xi) \leqslant \varepsilon$. ∎

5. Convergence of Iterates of a Contractive Self-Map

Proposition 4.3 and Theorem 4.6, taken together, provide another approach to the convergence of the iterates of a contractive self-map of $\mathbb{R}$; but Proposition 4.5 suggests that such an approach may not work for contractive maps in $\mathbb{R}^2$.

To deal with the convergence of the iterates of a contractive self-map on $\mathbb{R}^2$, we introduce the notion of an *approximate isometry*: a mapping f between metric spaces X, Y is an *ε-isometry* if $|\rho(f(x), f(x')) - \rho(x, x')| \leqslant \varepsilon$ for all x and x' in X. We shall need the following fundamental result about the extension of approximate isometries, whose proof we have placed later, in Section 6, in order not to interrupt the train of our work on fixed-points.

(5.1) Theorem. *Let B, ε be positive numbers, and n a positive integer. Then there exists $\delta > 0$ with the following property: if S is a set of n points of the closed ball with centre 0 and radius R in a Hilbert space H, and f is a nonexpansive map from the convex hull of S into H that is a δ-isometry on S, then f is an ε-isometry.*

The following trivial lemma is all we now need in preparation for our proof of the convergence of the iterates of a bounded contractive self-map of $\mathbb{R}^2$.

(5.2) Lemma. *Let $c, d_0, d_1, \ldots, d_k$ be real numbers with $c > d_0 - d_k$. Then there exists $i \in \{0, \ldots, k-1\}$ such that $d_i - d_{i+1} < c/k$.*

Proof. Since the sum $\sum_{i=0}^{k-1} (d_i - d_{i+1} - c/k)$ is negative, one of its terms is negative. ∎

(5.3) THEOREM. *If f is a bounded contractive self-map of $\mathbb{R}^2$, then the sequence (f^n) of iterates converges to the fixed point of f uniformly on each bounded subset of $\mathbb{R}^2$.*

Proof. Let ξ be the unique fixed point of f, r any positive number, and K the closed ball in $\mathbb{R}^2$ with centre ξ and radius r. In view of (2.5), we need only prove that for each $\alpha > 0$ and each x in K there exists n such that $\|f^n(x) - \xi\| \leqslant \alpha$. To this end, note that as K is totally bounded, there exist j, k such that $j < k$ and $\|f^j(x) - f^k(x)\| < \alpha$; so, replacing f by f^{k-j} and x by $f^j(x)$, we may assume that $\|x - f(x)\| < \alpha$. By (3.2), there exists $s > 0$ such that if $\|y - \xi\| \geqslant \alpha$, then $\|y - f(y)\| \geqslant s$. Construct a finite subset F of the circumference of the circle with centre ξ and radius $\alpha/2$, such that any angle with vertex ξ and size at least $s/2r$ includes a point of F. Let

$$\varepsilon \equiv \tfrac{1}{2} \inf\{\|z - \xi\| - \|f(z) - \xi\| : z \in F\} > 0.$$

By (5.1), there exists $\delta \in (0, s/2)$ such that if x_1, x_2, and x_3 are points of K, and f is a δ-isometry on $\{x_1, x_2, x_3\}$, then f is an ε-isometry of the convex hull of $\{x_1, x_2, x_3\}$.

Let n be an even integer greater than $6r/\delta$, and, in order to obtain a contradiction, suppose that $\|f^n(x) - \xi\| \geqslant \alpha$. For each i, let

$$r_i \equiv \|f^i(x) - \xi\|,$$
$$s_i \equiv \|f^i(x) - f^{i+1}(x)\|,$$
$$t_i \equiv r_i + s_i.$$

Then $r_0 > r_1 > \cdots$, $s_0 > s_1 > \cdots$, and $t_0 > t_1 > \cdots$. Also,

$$\begin{aligned} t_0 &= \|x - \xi\| + \|x - f(x)\| \\ &\leqslant \|x - \xi\| + \|f(x) - f(\xi)\| + \|x - \xi\| \\ &\leqslant 3\,\|x - \xi\| \leqslant 3r, \end{aligned}$$

so $t_0 - t_n \leqslant 3r < n\delta/2$. Applying (5.2) to the numbers $c \equiv n\delta/2$, $t_0, t_2, ..., t_n$, compute $m < n/2$ such that $t_m \leqslant t_{m+2} + \delta$. Then

$$\begin{aligned} (r_m - r_{m+1}) + (r_{m+1} - r_{m+2}) &= r_m - r_{m+2} \\ &\leqslant (r_m - r_{m+2}) + (s_m - s_{m+2}) \leqslant \delta, \end{aligned}$$

so $0 \leqslant r_m - r_{m+1} \leqslant \delta$ and $0 \leqslant r_{m+1} - r_{m+2} \leqslant \delta$; similarly, $0 \leqslant s_m - s_{m+1} \leqslant \delta$. Thus f is a δ-isometry on the set $S \equiv \{f^m(x), f^{m+1}(x), \xi\}$. By (5.1), f is an ε-isometry on the convex hull T of S. Let θ be the radian measure of the angle of T at the vertex ξ. Since $\|f^m(x)-\xi\| \geqslant \|f^n(x)-\xi\| \geqslant \alpha$, it follows from the choice of s, and elementary geometrical considerations, that

$$s \leqslant \|f^m(x)-f^{m+1}(x)\| \leqslant \theta r_m + (r_m - r_{m+1}) \leqslant \theta r + \delta$$

and therefore that $\theta \geqslant (s-\delta)/r > s/2r$. Thus T contains a point z of F. Since z is moved at most ε by the ε-isometry f on T, we have contradicted our choice of ε. This ensures that $\|f^n(x)-\xi\| \leqslant \alpha$, as we required. ∎

In attempting to extend this argument, to prove the convergence of a sequence of iterates of a bounded contractive self-map f of $\mathbb{R}^3$, the reader will appreciate the possibility of degeneracy to the $\mathbb{R}^2$ case, which creates difficulties that we have been unable to resolve.

Our discussion of fixed points raises several open questions:

1. *Is every bounded contractive self-map of* $\mathbb{R}^n$ *almost uniformly contractive?*

2. *Is every contractive self-map of a compact set* $K \subset \mathbb{R}^n$ *an almost uniform contraction?*

3. *Does every contractive self-map of a compact set* $K \subset \mathbb{R}^n$ *extend to a bounded contractive self-map of* $\mathbb{R}^n$*?*

4. *If f is a bounded self-map of* $\mathbb{R}^n$, *where* $n \geqslant 3$, *does the sequence of iterates of f converge pointwise to the unique fixed point of f?*

6. Nonexpansive Extensions of Approximate Isometries in a Hilbert Space

We end our paper with the postponed proof of Theorem 5.1. For this, we need some notations, definitions, and lemmas.

For points a, b of a Hilbert space, we write $a =_\delta b$ instead of $\|a-b\| \leqslant \delta$, and we use $[a, b]$ to denote the closed convex hull of $\{a, b\}$. We say that a mapping f is *ε-affine* on $[a, b]$ if

$$f(\lambda a + (1-\lambda)\, b) =_\varepsilon \lambda f(a) + (1-\lambda)\, f(b)$$

for each λ in $[0, 1]$. Thus f is a δ-isometry if and only if $\rho(f(x), f(y)) =_\delta \rho(x, y)$ for all x, y in the domain of f.

(6.1) Lemma. *Let u, v be elements of a Hilbert space H, $r \geqslant \max\{1, \|u\|, \|v\|\}$, $0 \leqslant \delta \leqslant 1$, and $0 \leqslant \lambda \leqslant 1$. If $\|u\| =_\delta \lambda\, \|u+v\|$ and $\|v\| =_\delta (1-\lambda)\, \|u+v\|$, then* $u =_{5\sqrt{r\delta}} \lambda(u+v)$.

Proof. Given $\varepsilon > \sqrt{r\delta}$, we need only show that $\|u - \lambda(u+v)\| \leqslant 5\varepsilon$. Either $\|u+v\| > 0$ or else $\|u+v\| < \frac{1}{2}(5\varepsilon - \delta)$. In the latter case,

$$\begin{aligned}\|u-\lambda(u+v)\| &\leqslant \|u\| + \lambda\ \|u+v\| \\ &\leqslant 2\lambda\ \|u+v\| + \delta \qquad (\text{as } u =_\delta \lambda(u+v)) \\ &\leqslant 5\varepsilon.\end{aligned}$$

So we may assume that $\|u+v\| > 0$.

Let π be the projection of H on the line through 0 and $u+v$. Noting that $\|v\| + \|v\| =_{2\delta} \|u+v\|$, we have

$$\|u\| + \|v\| - 2\delta \leqslant \|u+v\| = \|\pi u + \pi v\| \leqslant \|\pi u\| + \|v\|;$$

whence $\|u\| - \|\pi u\| \leqslant \delta$. As $\|u - \pi u\|^2 + \|\pi u\|^2 = \|u\|^2$, we have

$$\|u - \pi u\|^2 = (\|u\| - \|\pi u\|)(\|u\| + \|\pi u\|) \leqslant (2\delta)(2r) = 4\varepsilon^2,$$

so $u =_{2\varepsilon} \pi u$.

Now choose t such that $\pi u = t(u+v)$. Then $u =_{2\varepsilon} t(u+v)$ and therefore $v =_{2\varepsilon} (1-t)(u+v)$. From the first of these inequalities we see that

$$|t| =_{2\varepsilon/\|u+v\|} \|u\|/\|u+v\| =_{\delta/\|u+v\|} \lambda,$$

so $|t| =_{3\varepsilon/\|u+v\|} \lambda$. Thus if $t > 0$, then

$$t =_{3\varepsilon/\|u+v\|} \lambda. \tag{6.1.1}$$

Similarly, $|1-t| =_{3\varepsilon/\|u+v\|} (1-\lambda)$, so (6.1.1) holds if $t < 1$. Since either $t > 0$ or $t < 1$, it follows that

$$u =_{2\varepsilon} t(u+v) =_{3\varepsilon} \lambda(u+v)$$

and therefore that $u =_{5\varepsilon} \lambda(u+v)$. Hence $\|u - \lambda(u+v)\| \leqslant 5\varepsilon$. ∎

(6.2) LEMMA. *Let a, b be points in a Hilbert space H, $r \geqslant \max\{1, \|a-b\|\}$, $\delta > 0$, and $\varepsilon \equiv 5\sqrt{\delta r}$. Let f: $[a, b] \to H$ be a nonexpansive mapping that is a δ-isometry on $\{a, b\}$. Then f is a δ-isometry, and an ε-affine map, on $[a, b]$.*

Proof. Let $x \equiv \lambda a + (1-\lambda) b$, where $0 \leqslant \lambda \leqslant 1$. We have

$$\begin{aligned}\|f(a) - f(x)\| &\leqslant \|a - x\| \\ &= (1-\lambda)\ \|a-b\| \\ &\leqslant (1-\lambda)(\|f(a)-f(b)\| + \delta) \quad (\text{as } f \text{ is a } \delta\text{-isometry on } \{a, b\}) \\ &\leqslant (1-\lambda)\ \|f(a) - f(b)\| + \delta.\end{aligned}$$

Also,

$$\|f(b)-f(x)\| \leqslant \|b-x\| = \lambda \|a-b\|,$$

so that

$$\begin{aligned} \|f(a)-f(x)\| &\geqslant \|f(a)-f(b)\| - \|f(b)-f(x)\| \\ &\geqslant \|a-b\| - \delta - \lambda \|a-b\| \\ &= (1-\lambda) \|a-b\| - \delta \\ &\geqslant (1-\lambda) \|f(a)-f(b)\| - \delta. \end{aligned}$$

Hence $\|f(a)-f(x)\| =_\delta (1-\lambda) \|f(a)-f(b)\|$. Similarly, $\|f(b)-f(x)\| =_\delta \lambda \|f(a)-f(b)\|$. Applying Lemma 6.1 with $u \equiv f(a)-f(x)$ and $v \equiv f(x)-f(b)$, we now see that

$$f(a)-f(x) =_\varepsilon (1-\lambda)(f(a)-f(b);$$

whence

$$f(x) =_\varepsilon \lambda f(a) + (1-\lambda) f(b).$$

Now consider any two points x and y of $[a, b]$. Choose λ and μ in $[0, 1]$ such that $x \equiv \lambda a + (1-\lambda) b$ and $y \equiv \mu a + (1-\mu) b$. We have

$$\begin{aligned} \|a-b\| - \delta &\leqslant \|f(a)-f(b)\| \\ &\leqslant \|f(a)-f(x)\| + \|f(x)-f(y)\| + \|f(y)-f(b)\| \\ &\leqslant \|a-x\| + \|f(x)-f(y)\| + \|y-b\| \\ &= \|a-b\| + (\mu-\lambda) \|a-b\| + \|f(x)-f(y)\|, \end{aligned}$$

so

$$\|f(x)-f(y)\| \geqslant (\lambda-\mu) \|a-b\| - \delta.$$

Interchanging the roles of x and y in this argument, we see that

$$\|f(x)-f(y)\| \geqslant (\mu-\lambda) \|a-b\| - \delta;$$

whence

$$\|f(x)-f(y)\| \geqslant |\lambda-\mu| \|a-b\| - \delta.$$

Since also $\|f(x)-f(y)\| \leqslant \|x-y\|$, it follows that $f(x)-f(y) =_\delta x-y$. ∎

(6.3) LEMMA. *For all positive R and ε there exists $\delta > 0$ such that if a_1, a_2, a_3 are points of a Hilbert space H, each with norm at most R, and*

f: $[a_1, a_2] \cup \{a_3\} \to H$ is a nonexpansive map that is a δ-isometry on $\{a_1, a_2, a_3\}$, then f is an ε-isometry.

Proof. Let B be the closed ball in H with centre 0 and radius R. From the equation

$$\|\lambda u + \mu v\|^2 = \lambda^2 \|u\|^2 + \mu^2 \|v\|^2 + \lambda\mu(\|u\|^2 + \|v\|^2 - \|u - v\|^2),$$

which is valid for all λ, μ in $[0, 1]$ and all u, v in H, we readily see that there exists $\delta_1 > 0$, depending on R and ε but not on the values λ, μ in $[0, 1]$, such that if u, u', v, and v' are points of B with $|\ \|u\| - \|u'\|\ | \leqslant \delta_1$ and $|\ \|v\| - \|v'\|\ | \leqslant \delta_1$, then

$$\|\lambda u + \mu v\| \leqslant \|\lambda u' + \mu v'\| + \varepsilon/2.$$

On the other hand, by Lemma 6.2, there exists $\delta_2 > 0$ such that if a, b belong to B, and f: $[a, b] \to H$ is a nonexpansive mapping that is a δ_1-isometry on $\{a, b\}$, then f is an $\varepsilon/2$-affine map on $[a, b]$. Let $\delta \equiv \min\{\delta_1, \delta_2\}$. Consider points a_1, a_2, a_3 of B, and a nonexpansive map f: $[a_1, a_2] \cup \{a_3\} \to H$ that is a δ-isometry on $\{a_1, a_2, a_3\}$. Let $x \equiv \lambda a_1 + (1 - \lambda) a_2$, where $0 \leqslant \lambda \leqslant 1$. We have

$$\|f(x) - \lambda f(a_1) - (1 - \lambda) f(a_2)\| \leqslant \varepsilon/2,$$

by our choice of δ_2; also,

$$\|a_i - a_j\| - \delta_1 \leqslant \|f(a_i) - f(a_j)\| \leqslant \|a_i - a_j\| \qquad (1 \leqslant i, j \leqslant 3).$$

So

$$\begin{aligned}\|f(x) - f(a_3)\| &\geqslant \|\lambda(f(a_1) - f(a_3)) + (1 - \lambda)(f(a_2) - f(a_3))\| - \varepsilon/2 \\ &\geqslant \|\lambda(a_1 - a_2) + (1 - \lambda)(a_2 - a_3)\| - \varepsilon.\end{aligned}$$

As f is nonexpansive, it follows that f is an ε-isometry on $[a_1, a_2] \cup \{a_3\}$. ∎

We are now ready to give the Proof of Theorem 5.1.

Let B, ε be positive numbers, and n a positive integer. Using Lemma 6.3, we can find positive numbers $\delta_n \equiv \varepsilon > \delta_{n-1} > \cdots > \delta_1$ such that if H is a Hilbert space, A is any subset of the closed ball with centre 0 and radius R in H, and f is a nonexpansive map from the convex hull of A to H that is a δ_k-isometry on A, then f is a δ_{k+1}-isometry on $\bigcup_{a,b \in A} [a, b]$. Let $\delta \equiv \delta_1$, let $S \equiv \{a_1, ..., a_n\}$ be a subset of B, and for $k = 1, ..., n - 1$ let C_k be the convex hull of $\{a_1, ..., a_k\}$. Since

$$C_{k+1} = \cup\{[a_k, x] : x \in C_k\},$$

we see that if f is a nonexpansive map from the convex hull of S into H that is a δ-isometry on S, then f is an ε-isometry. ▮

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